

Green's function method and the first-order linear differential equation

Mário N. Berberan-Santos

Received: 1 February 2010 / Accepted: 31 March 2010 / Published online: 11 April 2010
© Springer Science+Business Media, LLC 2010

Abstract Use of a compact form of the general solution of the first-order linear differential equation allows establishing a direct connection with the Green's function method, providing an interesting interpretation of the result. This interpretation is valid for a wide range of physical and chemical problems.

Keywords First-order linear differential equation · Green's function · Linear response

1 Introduction

The general solution of the first-order linear differential equation is well-known, however, it is given in textbooks in an unnecessarily complicated and physically opaque form. A small change leading to a more compact form allows establishing a direct connection with the Green's function method, providing an interesting interpretation of the result that is relevant for many scientific and technological areas.

The general solution of the first-order linear differential equation

$$y' + P(x)y = Q(x), \quad (1)$$

M. N. Berberan-Santos (✉)
Centro de Química-Física Molecular and Institute of Nanoscience and Nanotechnology,
Instituto Superior Técnico, Universidade Técnica de Lisboa, 1049-001 Lisboa, Portugal
e-mail: berberan@ist.utl.pt

first obtained by Leibniz [1], is usually given in textbooks [1–3] as

$$f(x) = f(a) \exp\left(-\int_a^x P(u)du\right) + \exp\left(-\int_a^x P(u)du\right) \int_a^x Q(t) \exp\left(\int_a^t P(u)du\right) dt, \quad (2)$$

where the first term is the general solution of the homogeneous equation and the second term is a particular solution to the inhomogeneous equation.

It is nevertheless possible to rewrite the solution in a more compact form where both $P(x)$ and $Q(x)$ appear only once [4],

$$f(x) = \int_a^x [f(a)\delta(t-a) + Q(t)] \exp\left(-\int_t^x P(u)du\right) dt. \quad (3)$$

2 Connection with the Green's function method

Equation (3) admits a physically interesting interpretation within the linear response formalism, in which both $P(x)$ and $Q(x)$ are always non-negative functions, and the independent variable x is the time. In fact, Eq. (1) can be rewritten as

$$y' = -P(x)y + Q(x), \quad (4)$$

and defines the rate of change of a physical quantity y as a linear function of y , whose coefficients are two functions $P(x)$ and $Q(x)$. This equation applies to many physical situations. Within this linear response formalism, $Q(x)$ is called the impulse function. The magnitude of $Q(x)$ defines the degree of response of the system to a given external stimulus, whereas the functional (time) dependence of $Q(x)$ is externally controlled, sometimes at will. The stimulus can take place at a “single instant” (delta impulse) or may occur during an extended period, in a periodic (e.g. sinusoidal) or aperiodic way. The function $P(x)$ is on the other hand an intrinsic characteristic of the system defining its response to the external impulses. If $P(x)$ is non-negative, this response takes the form of a relaxation, i.e., a return to the initial state.

The solution of the associated differential equation

$$y' + P(x)y = \delta(x - x_0), \quad (5)$$

satisfying the condition $y(x) = 0$ for $x < x_0$, is the so-called Green's function [5] that can be interpreted as the response of the physical system to an impulse taking place at x_0 ,

$$G(x, x_0) = H(x - x_0) \exp\left(-\int_{x_0}^x P(u)du\right), \tag{6}$$

where $H(x)$ is Heaviside’s function (unit step function). The fact that $P(x)$ is non-negative ensures that the impulse has a transient effect only: The Green’s function represents a monotonically decaying response to a delta impulse that occurs at time x_0 . The form of $P(x)$ defines the exact shape of the time response. In the simplest case this function reduces to a positive constant k , and the decay is exponential.

If the impulse function (which is non-negative) has the general form $Q(x)$, then the system’s response is given by Eq. (3) that can be rewritten as

$$f(x) = \int_{x_0}^x [f(x_0)\delta(t - x_0) + Q(t)] G(x, t)dt \quad (x \geq x_0), \tag{7}$$

or as

$$f(x) = f(x_0)G(x, x_0) + \int_{x_0}^x Q(t)G(x, t)dt \quad (x \geq x_0). \tag{8}$$

The full response at time x is thus the sum of two positive terms. The first term results from the state of the system at time x_0 , before any impulse (as contained in $Q(x)$) has had the time to act. Even if no impulse is generated, the system fades away from the initial value $f(x_0)$ by internal relaxation processes. This initial value may have resulted from impulses acting before x_0 , but the knowledge of their detailed form is not necessary. All that is required is the value of $f(x)$ for x_0 , and not how it was arrived at. As a particular case, the system’s initial value $f(x_0)$ may even result from a delta impulse $f(x_0)\delta(t - x_0)$, as follows from Eq. (7). Finally, the second term of Eq. (8) accounts for all subsequent delta impulse responses, each generated at a different time $x_0 < t \leq x$, and each with initial value $Q(t)$.

3 Concluding remarks

The Green’s function method as applied to the first-order linear differential equation differs from its usual application to second-order differential equations. In fact, a single condition needs to be specified, and the problem is not a boundary value one as usual, e.g. in Sturm-Liouville type equations [5]. This one-sidedness creates an essential asymmetry with respect to the independent variable (time) allowing causality to be respected, provided the proper choice of signs in $P(x)$ and $Q(x)$ and of the initial condition in the Green’s function derivation is made.

In conclusion, use of a compact form of the general solution of the first-order linear differential equation allowed establishing a direct connection with the Green’s function method, providing an interesting interpretation of the result. This interpretation

is valid for a wide range of physical and chemical problems, e.g. in chemical kinetics [4,6] and in time-resolved spectroscopy [7].

References

1. E.L. Ince, *Ordinary Differential Equations*, Dover, New York. (1956), (reprint of the 1926 edition), pp. 20–21
2. T.A. Apostol, *Calculus*, 2nd edn., vol. 1. (Blaisdell, Waltham, 1967), pp. 309–311
3. D.A. McQuarrie, *Mathematical Methods for Scientists and Engineers*. (University Science Books, Sausalito, 2003), pp. 524–526
4. M.N. Berberan-Santos, *J. Math. Chem.* **47**, 1184 (2010)
5. D.G. Duffy, *Green's Functions with Applications* (Chapman & Hall/CRC, Boca Raton, 2001), p. 18
6. M.N. Berberan-Santos, *MATCH Comm. Math. Comp. Chem.* **63**, 603 (2010)
7. B. Valeur, *Molecular Fluorescence* (Wiley-VCH, Weinheim, 2002), p. 167